FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING--X: RESOLVENTS, CHARACTERISTIC FUNCTIONS AND VALUES

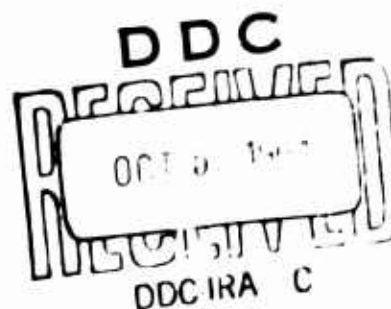Richard Bellman
Sherman Lehman

P-1332

Revised November 14, 1958

SUMMARY

In previous papers it has been shown how the functional
equation technique of dynamic programming could be applied to
derive variational relations for Green's functions. In this
paper we apply the method to the variational problem yielding
the equation

$$(pu')' + (r(x) + \lambda q(x))u = v(x), \quad u(a) = u(1) = 0.$$

The introduction of the parameter λ enables us to study the
resolvent operator, and thus to derive variational relations
for the characteristic values and the characteristic functions
of the associated Sturm—Liouville equation.

FUNCTIONAL EQUATIONS IN THE THEORY
OF DYNAMIC PROGRAMMING—X:
RESOLVENTS, CHARACTERISTIC FUNCTIONS AND VALUES

Richard Bellman
Sherman Lehman

1. Introduction

In previous papers, we have shown how the functional
equation technique of dynamic programming can be applied to
derive variational relations for kernels and Green's functions.
In [1], the Green's function associated with the second order
equation

(1) $u'' + q(x)u = 0, \quad u(a) = u(1) = 0,$

was discussed, while in [4], analogous methods were applied to
partial differential operators to obtain the Hadamard varia-
tional formula. In [2], the Fredholm integral equation was
treated by similar means, and Jacobi matrices were discussed
in [3].

In this paper, we wish to present some
extensions of these results. Introducing the parameter λ,
we consider the general equations

(2) $(p(x)u')' + (r(x) + \lambda q(x))u = 0, \quad u(a) = 0, \quad u(1) = 0,$

obtaining, as in the papers cited above, variational equations
for the resolvent operator as a function of a. Utilizing the
meromorphic nature of the operator as a function of λ, we are
able in this way to derive variational equations for the

characteristic values and functions.

Corresponding results are obtained for the vector—matrix system

(3) $(P(t)x')' + \lambda Q(t)x = 0.$

In both cases, certain assumptions have to be made con—cerning the coefficient functions, $q(x)$ and $Q(t)$, in order to be able to consider the differential equations as the Euler equations of associated variational problems. Since, however, we know, from the results of Miller and Schiffer, [7], con—cerning Green's functions for general linear differential operators of order n, (where quite different methods are employed) and from corresponding results for Fredholm operators and Jacobi matrices, that the relations obtained hold under far weaker assumptions, the interesting problem arises of deriving the more general results by variational techniques. In this paper, we present a method of analytic continuation which reduces the case of continuous $q(x)$ to that of positive continuous $q(x)$, and the case of continuous symmetric $Q(t)$ to positive definite $Q(t)$.

In a separate paper, [5], we have sketched extensions of this technique of analytic continuation which enable us to treat non—selfadjoint differential operators and non—symmetric matrices by means of variational methods.

Finally, in a brief section at the end of the paper, we indicate how similar methods may be applied to obtain corres—ponding results for Fredholm kernels and the associated

characteristic values and functions.

2. Variational Problem

Consider the boundary value problem

(1) $(p(x)u')' + q(x)u = v(x), \quad a < x < 1,$

$$u(a) = 0, \quad u(1) + au'(1) = 0.$$

We shall assume that p, q, and v are continuous functions on the closed interval $[a,1]$. If the corresponding Sturm-Liouville problem with q replaced by λq and v by 0 does not have 1 as a characteristic value, then the unique solution of (1) can be represented in the form

(2) $u(x) = \int_a^1 K(x,y,a)v(y)dy.$

The function $K(x,y,a)$ is called the Green's function for the boundary value problem (1). We wish to study the dependence of K upon a by means of the functional equation method of dynamic programming.

We imbed (1) in the system

(3) $(p(x)u')' + q(x)u = v(x), \quad a < x < 1,$

$$u(a) = c, \quad u(1) + au'(1) = 0.$$

A solution of this system can be expressed in the form

(4) $u(x) = u_0(x) + c\phi(x)$

where $u_0(x)$ is a solution of (1) and hence equal to the right side of (2) while $\phi(x)$ is a solution of the system

(5) $\qquad (p(x)\phi')' + q(x)\phi = 0, \quad \phi(a) = 1, \quad \phi(1) + a\phi'(1) = 0.$

We also consider a variational problem associated with the system (3), the problem of maximizing $J(u,v)$ over all u for which $u(a) = c$ where

(6) $\qquad J(u,v) = \int_a^1 (q(x)u^2 - p(x)u'^2 - 2uv(x))dx$

$$- \frac{p(1)}{a} (u(1))^2.$$

This variational problem has the property that a u yielding the maximum is a solution of the system (3). To prove this, we replace u by $u + \epsilon\eta$ in (6), where $\eta(x)$ is a function such that $\eta(a) = 0$. We obtain

$$J(u + \epsilon\eta, v) = J(u,v) + 2\epsilon\left\{ \int_a^1 (qu\eta - pu'\eta' - v\eta)dx \right.$$

$$\left. - \frac{p(1)}{a} u(1)\eta(1) \right\}$$

$$+ \epsilon^2\left[\int_a^1 (q\eta^2 - p\eta'^2)dx - \frac{p(1)}{a} \eta(1)^2 \right].$$

In order to have u yield the maximum, the ϵ term must vanish for all functions η for which $\eta(a) = 0$. Hence

$$\int_a^1 (qu + (pu')' - v)\eta(x)dx - \frac{p(1)}{a} \eta(1)(u(1) + au'(1)) = 0.$$

Consequently, a u yielding the maximum must satisfy the
boundary condition at 1 and must satisfy the differential
equation in the interior of the interval $[a,1]$. Hence, a
solution of the variational problem provides a solution of the
boundary value problem (3).

In order to use the variational approach one must make an
assumption on p and q sufficient to guarantee the existence
of a maximum. It is sufficient, for example, to assume that
$p(x)$ is positive on the closed interval $[a,1]$ and that the
smallest characteristic value of the Sturm—Liouville prob em

$$(pu')' + \lambda qu = 0, \quad u(a) = 0, \quad u(1) + \alpha u'(1) = 0$$

is larger than 1. If $q(x)$ is uniformly positive,
$q(x) \geq d > 0$, over $0 \leq x \leq a$, this condition holds if a is
sufficiently small, or if d is sufficiently large. This
assumption guarantees the existence of a unique maximum of
$J(u,v)$. In §7, we shall show by analytic continuation how the
results we prove can be freed of this restrictive hypothesis.
At the moment we want $q(x)$ to be positive so that we can easily
locate a region within which no characteristic values occur.

3. Dynamic Programming Approach

We now let

$$f(a,c) = \max_{u(a)=c} J(u,v)$$

and derive a partial differential equation for f by means of
a technique of dynamic programming. We regard u as describing

a policy. The variable c describes the state of the system at a. The result of following the policy u for a time interval $[a, a + \Delta]$ is to transform c into a new initial state $u(a + \Delta)$ for the interval $[a + \Delta, 1]$. According to the principle of optimality, an optimal policy u over the interval $[a, 1]$ with initial state c must have the property that it is an optimal policy over the subinterval $[a + \Delta, 1]$ starting with initial state $u(a + \Delta)$. Translated into a formula, the principle of optimality yields the equation

$$f(a, c) = \max_{u(a) = c} \left\{ f((a + \Delta), u(a + \Delta)) \right.$$

$$\left. + \int_a^{a+\Delta} (q(x)u(x)^2 - p(x)u'(x)^2 - 2u(x)v(x))dx \right\}.$$

We proceed formally, assuming that $f(a, c)$ has continuous partial derivatives and that the maximizing u has a continuous derivative. Then as $\Delta \to 0$,

$$f(a + \Delta, u(a + \Delta)) = f(a, c) + \frac{\partial f}{\partial a}(a, c)$$

$$+ \frac{\partial f}{\partial c}(a, c)u'(a)\Delta + o(\Delta).$$

Consequently, as $\Delta \to 0$, we have

$$-\frac{\partial f}{\partial a}(a, c) = \max_{u(a) = c} \left\{ \frac{\partial f}{\partial c}(a, c)u'(a) + q(a)c^2 \right.$$

$$\left. - p(a)u'(a)^2 - 2cv(a) + o(1) \right\}.$$

The quadratic

$$\frac{\partial f}{\partial c} u' - p(a)u'^2$$

takes its maximum value at

$$u' = \frac{1}{2p(a)} \frac{\partial f}{\partial c} (a,c).$$

Hence we obtain the partial differential equation

(1) $$-\frac{\partial f}{\partial a} (a,c) = \frac{1}{4p(a)} (\frac{\partial f}{\partial c}(a,c))^2 - 2cv(a) + c^2q(a).$$

4. Variation of the Green's Function

Let u be the function which maximizes $J(u,v)$ for given a and c. By using (2.2) and (2.4), we can find an equation which connects $f(a,c)$ with the Green's function $K(x,y,a)$. We have

(1) $$f(a,c) = \int_a^1 (qu^2 - pu'^2 - 2uv)dx - \frac{p(1)}{a} (u(1))^2$$

$$= \int_a^1 u(qu + (pu')' - v)dx - \int_a^1 uvdx - u(1)p(1)u'(1)$$

$$+ u(a)p(a)u'(a) - \frac{p(1)}{a} (u(1))^2$$

$$= -\int_a^1 uvdx + cp(a)u'(a) - \frac{p(1)}{a} u(1)(u(1) + au'(1))$$

$$= -\int_a^1 u_0vdx - 2\int_a^1 \emptyset vdx + cp(a)u_0'(a) + c^2p(a)\emptyset'(a).$$

Using the fact that u_0 satisfies (2.1) and $\emptyset$ satisfies (2.6), we obtain by integration by parts

$$\int_a^1 \phi v \, dx = \int_a^1 \phi((pu_0')' + qu_0) \, dx$$

$$= \int_a^1 u_0((p\phi')' + q\phi) \, dx + \phi(1)p(1)u_0'(1)$$

$$- \phi(a)p(a)u_0'(a)$$

$$- \phi'(1)p(1)u_0(1)$$

$$+ \phi'(a)p(a)u_0(a).$$

Since $u_0(a) = 0$, $\phi(a) = 1$, and u and ϕ satisfy the same homogeneous boundary condition at 1, we conclude that

(2) $$\int_a^1 \phi v \, dx = -p(a)u_0'(a) + (\phi(1)p(1)u_0'(1) - \phi'(1)p(1)u_0(1))$$

$$= -p(a)u_0'(a).$$

Thus, the second and third terms on the last line of (1) are equal. It turns out to be more convenient to use the integral expression. We have

(3) $$f(a,c) = -\int_a^1 u_0 v \, dx - 2c\int_a^1 \phi v \, dx + c^2 p(a)\phi'(a)$$

$$= -\int_a^1\int_a^1 K(x,y)v(x)v(y) \, dx \, dy - 2c\int_a^1 \phi v \, dx + c^2 p(a)\phi'(a).$$

We observe that we can express ϕ in terms of a partial derivative of K. By (2), we have

$$\int_a^1 \phi(y)v(y)dy = - p(a)u_a(\cdot) = - r(a) \int_a^1 \frac{\partial K}{\partial X}(a,y,a)v(y)dy$$

for all continuous functions v. Thus

(4) $\qquad \phi(y) = - p(a) \frac{\partial K}{\partial X}(a,y,a).$

Combining the expression for $\phi(x,a)$ given by (4) with
the partial differential equation (3.1), we obtain upon
equating terms independent of a,

$$\int_a^1 \int_a^1 \frac{\partial K}{\partial a}(x,y,a)v(x)v(y)dx dy = \frac{1}{4p(a)} \int_a^1 \phi(x)v(x)dx$$

$$= \frac{1}{r(a)} \int_a^1 \int_a^1 \phi(x)\phi(y)v(x)v(y)dx dy$$

Now if we "equate coefficients" of $v(x)v(y)$ we obtain the
relation

(5) $\qquad \frac{\partial K}{\partial a}(x,y,a) = \frac{1}{p(a)} \phi(x)\phi(y) = p(a) \frac{\partial K}{\partial X}(x,a,a) \frac{\partial K}{\partial X}(a,y,a).$

This formal equating of coefficients can be justified with the
use of the symmetry of the Green's function and an assumption
such as continuity of

$$\frac{\partial K}{\partial a}(x,y,a) - \frac{1}{r(a)} \phi(x)\phi(y).$$

We shall prove this in 5.

In the consideration of the boundary value problem (2.3),
we have not considered the boundary conditions $u(1) = 0$ or

$u'(1) = 0$. The condition $u(1) = 0$ corresponds formally to $\alpha = 0$; the condition $u'(1) = 0$ corresponds formally to $\alpha = \infty$ in the condition $u(1) + \alpha u'(1) = 0$. If in the variational problem of maximizing $J(u,v)$ we omit the term

$$-\frac{r(1)}{\alpha} u(1)^2$$

and add the constraint $u(1) = 0$ or $u'(1) = 0$, respectively, the development proceeds as before, except all terms which involve α vanish. The final results obtained are the same.

5. Justification of Equating Coefficients

The formal procedure used in the derivation of (4.5) can be justified by the following lemma.

Lemma. Let $F(x,y)$ be a continuous function on the region $a < x < 1$, $a < y < 1$ and suppose that $F(x,y) = F(y,x)$. Then if

$$\int_a^1 \int_a^1 F(x,y)v(x)v(y)dxdy = 0$$

for all continuous functions v, the function $F(x,y) \equiv 0$.

Proof. First, we show that F vanishes on the diagonal, i.e., $F(\xi,\xi) = 0$. To do this, we take a sequence of continuous functions v_ϵ with v positive and tending to 0, each of which vanishes identically outside the interval $[\xi - \epsilon, \xi + \epsilon]$ and integrates to 1 over the portion of $[\xi - \epsilon, \xi + \epsilon]$ which is in the interval $[a,1]$. Passing to the limit, we find that $F(\xi,\xi) = 0$.

To prove that $F(\zeta,\eta) = 0$ when $\zeta \neq \eta$ we use functions v_ϵ with $\epsilon < |\zeta - \eta|/2$ which integrate to 1 over the portion of $[\zeta - \epsilon, \zeta + \epsilon]$ in the interval $[a,1]$ and also integrate to 1 over the portion of $[\eta - \epsilon, \eta + \epsilon]$ in $[a,1]$ but which vanish identically outside these two intervals. Letting $\epsilon \to 0$, we find

$$F(\zeta,\eta) + F(\eta,\zeta) + F(\zeta,\zeta) + F(\eta,\eta) = 0.$$

Since $F(\zeta,\zeta) = F(\eta,\eta) = 0$ and $F(\zeta,\eta) = F(\eta,\zeta)$, we have

$$F(\zeta,\eta) = 0.$$

6. Change of Variable

In this section, we shall make a change of variable which leads to a different expression for $\phi(x)$ and hence an alternate expression for the variation of the Green's function. We make the change of variable

$$u = c\frac{(1 + \alpha - x)}{(1 + \alpha - a)} + w$$

so that $w(a) = 0$ and $w(1) + \alpha w'(1) = 0$ when u satisfies the boundary conditions $u(a) = c$, $u(1) + \alpha u'(1) = 0$. (This transformation is valid also in case $\alpha = 0$; for the condition $u'(1) = 0$, i.e., $\alpha = \infty$, we set $u = c + w$).

Then by (2.6),

$$J(u,v) = J(w,v) + 2c \int_a^1 \left[\frac{1 + \alpha - x}{1 + \alpha - a}\right] q(x)w(x)dx$$

$$+ 2 \int_a^1 \frac{p(x)w'(x)}{1 + \alpha - a} \, dx - 2c \int_a^1 v(x) \left[\frac{1 + \alpha - x}{1 + \alpha - a}\right] \, dx$$

$$- 2c \frac{p(1)w(1)}{1 + \alpha - a}$$

$$+ c^2 \left\{ \int_a^1 \left[\frac{1 + \alpha - x}{1 + \alpha - a}\right]^2 q(x)dx - \int_a^1 \frac{p(x)}{(1 + \alpha - a)^2} \, dx \right.$$

$$\left. - \frac{p(1)\alpha}{(1 + \alpha - a)^2} \right\} .$$

Transforming the second integral by integration by parts, we obtain

$$J(u,v) = J\left\{w, v - c\left[\left(\frac{1 + \alpha - x}{1 + \alpha - a}\right)q(x) - \frac{p'(x)}{1 + \alpha - a}\right]\right\} + F(a,c,v)$$

where F is independent of w and u. Hence

$$f(a,c) = \underset{u(a)=}{\text{Max}} \quad J(u,v) = \underset{w(a)=0}{\text{Max}} \quad J\left\{w, v - c\left[\left(\frac{1 + \alpha - x}{1 + \alpha - a}\right)q(x) - p'(x)\right]\right\}$$

$$+ F(a,c,v).$$

The maximizing w is given by

$$w = \int_a^1 K(x,y,a)\left\{v(y) - c\left[\left(\frac{1 + \alpha - y}{1 + \alpha - a}\right)q(y) - \frac{p'(y)}{1 + \alpha - a}\right]\right\}dy.$$

Hence

$$u = \int_a^1 K(x,y,a)v(y)dy + c\left[\frac{1 + \alpha - x}{1 + \alpha - a}\right.$$

$$- \int_a^1 K(x,y,a)\left[\left(\frac{1 + \alpha - y}{1 + \alpha - a}\right) q(y) - \frac{p'(y)}{1 + \alpha - a}\right]dy\right].$$

Thus, by (2.4),

$$\phi(x) = \frac{1 + \alpha - x}{1 + \alpha - a} - \int_a^1 K(x,y,a)\left[\left(\frac{1 + \alpha - y}{1 + \alpha - a}\right)q(y) - \frac{p'(y)}{1 + \alpha - a}\right]dy.$$

Substituting this expression into (4.2), we find another formula for the variation of the Green's function,

$$(1) \quad \frac{\partial K(x,y,a)}{\partial a} = \frac{1}{p(a)}\left\{\frac{1 + \alpha - x}{1 + \alpha - a} - \int_a^1 K(x,y,a)\left[\left(\frac{1 + \alpha - y}{1 + \alpha - a}\right)q(y)\right.\right.$$

$$\left.- \frac{p'(y)}{1 + \alpha - a}\right]dy\right\}$$

$$\cdot \left\{\frac{1 + \alpha - y}{1 + \alpha - a} - \int_a^1 K(x,y,a)\left[\left(\frac{1 + \alpha - x}{1 + \alpha - a}\right)q(y)\right.\right.$$

$$\left.- \frac{p'(y)}{1 + \alpha - a}\right]dy\right\}.$$

The cases $u(1) = 0$ and $u'(1) = 0$ are easily handled. The results are what would be obtained by putting $\alpha = 0$ or letting $\alpha \to \infty$ respectively.

7. Analytic Continuation

In order to extend the foregoing approach to the general case, we shall employ the method of analytic continuation.

Consider the equation

(1) $(pu')' + (z + q(x))u = v,$

           $u(a) = c,\quad u(1) + \alpha u'(1) = 0,$

where z is a positive quantity chosen large enough so that
$z + q(x) \geq d > 0$ in $\left[a,1\right]$.

Furthermore, we know from the Sturmian comparison theorems
that z can be chosen large enough so that $\lambda = 1$ is not a
characteristic value of the Sturm-Liouville problem

(2) $(pu')' + \lambda(z + q(x))u = 0,$

           $u(a) = 0,\quad u(1) + \alpha u'(1) = 0.$

It follows that the inhomogeneous equation

(3) $(pu')' + (z + q(x))u = v,$

           $u(a) = 0,\quad u(1) + \alpha u'(1) = 0,$

will then have a unique solution.

The Green's function associated with this problem will be
a function of z. If we can show that this function is a
meromorphic function of z, it will follow that the relations
originally obtained under the assumption that z is
sufficiently large will actually be valid for all z distinct
from a set of poles.

In particular, the relations will be valid for $z = 0$,
provided that $\lambda = 1$ is not a characteristic value of the
equation

(4) $(pu')' + q(x)u = v,$

   $(a) = c,$ $u(1) + \alpha u'(1) = 0.$

Generally speaking, the relations will be valid whenever they make sense.

8. Analytic Character of Green's Function

Although it follows from well-known results in the theory of linear differential equations that the Green's function is a meromorphic function of z, we shall outline a proof here for the sake of completeness.

To simplify the notation let us write $\sigma(u) = 0$ for the boundary condition $u(1) + \alpha u'(1) = 0$. Furthermore we observe that the following considerations also cover the boundary condition $u'(1) = 0$, so that we could also take $\sigma(u) = u'(1)$.

To begin with, the two solutions of

(1) $(pu')' + (z + q(x))u = 0,$

determined by the initial conditions

(2) $u_1(a) = 1,$ $u_1'(a) = 0,$

   $u_2(a) = 0,$ $u_2'(a) = 1,$

are entire functions of z for x in $\left[0, 1\right]$. Also for fixed x, their derivatives u_1' and u_2' are entire functions of z. Hence, in particular, $\sigma(u_2)$ is an entire

function of z which does not vanish identically.

Concerning $p(x)$ and $q(x)$, we are assuming, as before, that $p(x) \geq d_1 > 0$ in $[0,1]$, that it has an integrable derivative, and that $q(x)$ is integrable in $[a,1]$, and uniformly bounded.

The general solution of

$$(3) \qquad (pu')' + (z + q(x))u = v$$

can be written

$$(4) \qquad u = c_1 u_1(x) + c_2 u_2(x) - \frac{1}{p(a)} \int_a^x \left[u_1(x) u_2(y) - u_1(y) u_2(x) \right] v(y) dy.$$

Imposing the boundary conditions

$$u(a) = 0, \quad \sigma(u) = 0,$$

we see that this particular solution has the form

$$u = \int_a^1 K(x,y) v(y) dy$$

where $K(x,y)$, the desired Green's function, is given by

$$K(x,y) = \frac{1}{p(a)} \left[\frac{\sigma(u_1) u_2(y) - u_1(y) \sigma(u_2)}{\sigma(u_2)} \right] u_2(x), \quad a < y < x,$$

$$= \frac{1}{p(a)} \left[\frac{\sigma(u_1) u_2(x) - u_1(x) \sigma(u_2)}{\sigma(u_2)} \right] u_2(y), \quad a < x < y.$$

We see that for any x and y in $[a,1]$, $K(x,y)$ and its partial derivatives with respect to x,y and a are mero-

morphic functions of z. Also, we observe that this formula provides a proof for the symmetry of the Green's function, a property which we have used in the previous sections.

9. Alternate Derivation of Expression for $\phi(x)$

The explicit representation for $K(x,y)$ given above shows that

$$\left.\frac{\partial K}{\partial x}\right|_{x=a} = \frac{1}{p(a)}\left[\frac{\sigma(u_1)u_2(y) - u_1(y)\sigma(u_2)}{\sigma(u_2)}\right].$$

It follows then that $-p(a)\frac{\partial K}{\partial x}(a,y,a)$ is the solution of the two-point boundary value problem

$$(pu')' + qu = 0, \quad u(a) = 1, \quad \sigma(u) = 0.$$

Hence we have

$$\phi(y) = -p(a)\frac{\partial K}{\partial x}(a,y,a).$$

The extension of the relation in (1) is the keystone of the treatment of Miller and Schiffer in [6], and the analogue of this relation for partial differential operators is essential in the derivation of the Hadamard variational formula given in [4].

10. Variation of Characteristic Values and Characteristic Functions

Consider the Sturm-Liouville problem

(1) $(pu')' + (q(x) + \lambda r(x))u = v(x), \quad a < x < 1,$

$u(a) = 0, \quad u(1) + \alpha u'(1) = 0.$

This problem has a unique solution, when λ is not a characteristic value, which is given by

$$u(x) = \int_a^1 R(x,y,\lambda,a)v(y)dy.$$

The function R, called the resolvent, is a meromorphic function of λ with poles at the characteristic values λ_1, λ_2, λ_3, ... If $\psi_k(x)$ is the characteristic function associated with the characteristic value λ_k, then R has the representation

$$(2) \quad R(x,y,\lambda,a) = \sum_{k=1}^{\infty} \frac{\psi_k(x)\psi_k(y)}{\lambda - \lambda_k} .$$

The problem (1) is obtained from (2.1) by replacing $q(x)$ by $q(x) + \lambda r(x)$. From (4.5) we obtain the equation

$$(3) \quad \frac{\partial R(x,y,\lambda,a)}{\partial a} = p(a) \frac{\partial R}{\partial x}(a,y,\lambda,a) \frac{\partial R}{\partial y}(x,a,\lambda,a).$$

This equation is valid for all $\lambda \neq \lambda_k$ without any assumptions on $q(x)$ and $r(x)$ except continuity as has been shown by analytic continuation in §7.

Combining (2) with (3), we find

$$\sum_{k=1}^{\infty} \left\{ \frac{\psi_k(x)\psi_k(y)}{(\lambda - \lambda_k)^2} \frac{\partial \lambda_k}{\partial a} + \frac{\psi_k(x) \frac{\partial \psi_k(y)}{\partial a} + \psi_k(y) \frac{\partial \psi_k(x)}{\partial a}}{\lambda - \lambda_k} \right\}$$

$$= p(a) \left(\sum_{k=1}^{\infty} \frac{\psi_k'(a)\psi_k(y)}{\lambda - \lambda_k} \right) \left(\sum_{k=1}^{\infty} \frac{\psi_k(x)\psi_k'(a)}{\lambda - \lambda_k} \right).$$

Letting $\lambda \to \lambda_k$ and equating coefficients of $(\lambda - \lambda_k)^{-2}$ we find

(4) $\quad \dfrac{\partial \lambda_k}{\partial a} = p(a)(\psi_k'(a))^2 .$

By equating coefficients of $(\lambda - \lambda_k)^{-1}$ we find

(5) $\quad \dfrac{\partial \psi_k(x)}{\partial a} = p(a)\psi_k'(a) \sum_{j \neq k} \dfrac{\psi_j(x)\psi_j'(a)}{\lambda - \lambda_j} .$

We can also derive expressions for the variation of the sums

$$ S_n(x,y,a) = \sum_{k=1}^{\infty} \dfrac{\psi_k(x)\psi_k(y)}{\lambda_k^n} , \quad (n = 1, 2, 3, \ldots). $$

Since

$$ \dfrac{1}{\lambda - \lambda_k} = - \dfrac{1}{\lambda_k} - \dfrac{\lambda}{\lambda_k^2} - \dfrac{\lambda^2}{\lambda_k^3} - \cdots , $$

we have

$$ R(x,y,\lambda,a) = - \sum_{n=1}^{\infty} S_n(x,y,a)\lambda^{n-1} . $$

Consequently,

$$ \dfrac{\partial S_n}{\partial a}(x,y,a) = - p(a) \sum_{\substack{j+k=n+1 \\ 1 \leq j \leq n \\ 1 \leq k \leq n}} \dfrac{\partial S_j(a,y,a)}{\partial x} \dfrac{\partial S_k(x,a,a)}{\partial y} . $$

11. Matrix Case

Analogous results can be obtained for a Green's function associated with the vector-matrix system

(1) $$\frac{d}{dt}(P(t)\frac{d}{dt}x(t)) + Q(t)x(t) = y(t), \quad a < t < 1,$$

$$x(a) = c, \quad \dot{x}(1) + Bx(1) = 0,$$

where P, Q, and B are $n \times n$ matrices and x, y, and c are n-dimensional vectors. We shall assume that P, Q, and y are continuous on the closed interval $[a,1]$ and that P(t) has an inverse for each t. Also we shall assume that P, Q, and P(1)B are symmetric matrices so that the system is self-adjoint.

For notation, we denote the transpose of a matrix A by A* and we use (u,v) to denote the inner product of two vectors u and v.

Associated with this boundary value problem there is the variational problem of maximizing J(x,y) where

$$J(x,y) = \int_a^1 \left\{(Qx,x) - (P\dot{x},x) - 2(y,x)\right\} dt$$

$$- (P(1)Bx(1),x(1))$$

subject to the condition that $x(a) = c$. Let

$$f(a,c) \quad \max_{x(a)=c} \quad J(x,y).$$

As in the scalar case, a vector x(t) which yields the maxi-

mum must be a solution of the boundary value problem (1).

Proceeding as in §3 one can use the principle of optimality from dynamic programming to derive the partial differential equation

$$(2) \qquad -\frac{\partial f}{\partial a}(a,c) = \frac{1}{4}(P^{-1}(a)\frac{\partial f}{\partial c},\frac{\partial f}{\partial c}) - 2(c,y(a)) + (Q(a)c,c)$$

where if $c_1, c_2, \ldots, c_n$ are the components of c then

$$\frac{\partial f}{\partial c} = \begin{pmatrix} \frac{\partial f}{\partial c_1} \\ \frac{\partial f}{\partial c_2} \\ \vdots \\ \frac{\partial f}{\partial c_n} \end{pmatrix}.$$

We wish to use this partial differential equation to study the Green's function for the system (1). The Green's function is an $n \times n$ matrix $K(t,s)$ for which

$$(3) \qquad x_0(t) = \int_a^1 K(t,s)y(s)ds$$

is a solution of (1) with $x_0(a) = 0$. As before a solution of (1) with $x(a) = c$ can be written as a sum

$$x(t) = x_0(t) + \phi(t)c$$

where $\phi(t)$ is the $n \times n$ matrix function which is the solution of

(4)
$$\frac{d}{dt}\left(P(t)\frac{d}{dt}\Phi(t)\right) + Q(t)\Phi(t) = 0,$$

$$\Phi(a) = I, \quad \dot{\Phi}(1) + B\Phi(1) = 0.$$

Since $x(t)$ is the maximizing vector for the variational problem, we obtain

(5)
$$f(a,c) = -\int_a^1 (y(t),x(t))dt - (P(1)B(1)x(1),x(1))$$

$$+ (P(a)\dot{x}(a),x(a)) - (P(1)\dot{x}(1),x(1))$$

$$= -\int_a^1 (y(t),x_0(t))dt - \int_a^1 (y(t),\Phi(t)c)dt$$

$$+ (P(a)x_0(a),c) + (P(a)\dot{\Phi}(a)c,c).$$

Also

(6)
$$\int_a^1 (y(t),\Phi(t)c)dt = \int_a^1 (x_0(t),\frac{d}{dt}P(t)\dot{\Phi}(t)c + Q\Phi c)dt$$

$$+ (P(1)\dot{x}_0(1),\Phi(1)c) - (P(a)\dot{x}_0(a),\Phi(a)c)$$

$$- (x_0(1),P(1)\dot{\Phi}(1)c) + (x_0(a),P(a)\dot{\Phi}(a)c)$$

$$= - (P(a)\dot{x}_0(a),c) - (P(1)Bx_0(1),\Phi(1)c)$$

$$+ (P(1)x_0(1),B\Phi(1)c)$$

$$= - (P(a)\dot{x}_0(a),c)$$

because of the assumption that $P(1)B$ is symmetric.

Hence by (3) and (5),

$$f(a,c) = - \int_a^1 \int_a^1 (K(t,s)y(s),y(t))dsdt$$

$$- 2\int_a^1 (y(t),\Phi(t)c)dt + (P(a)\Phi(a)c,c).$$

Also it is possible to express Φ in terms of a partial derivative of K. By (6),

$$\int_a^1 (c,\Phi^*(s)y(s))dt = - (c,P(a)\int_a^1 \frac{\partial K}{\partial t}(t,s,a)\Big|_{t=a} y(s)ds)$$

for all continuous vector functions y and all vectors c. Hence,

$$(7) \qquad \Phi^*(s) = - P(a)\frac{\partial K}{\partial t}(t,s,a)\Big|_{t=a}.$$

Next, we combine the above expression for $f(a,c)$ with the partial differential equation (2) and equate the terms independent of c to obtain

$$\int_a^1 \int_a^1 (\frac{\partial K(t,s,a)}{\partial a}y(s),y(t))dsdt$$

$$= (P^{-1}(a)\int_a^1 \Phi^*(t)y(t)dt, \int_a^1 \Phi^*(t)y(t)dt),$$

or

$$(8) \qquad \int_a^1 \int_a^1 \left(\left\{\frac{\partial K}{\partial a}(t,s,a) - \Phi(t)P^{-1}(a)\Phi^*(s)\right\} y(s),y(t)\right)dsdt = 0.$$

Now we "equate coefficients" to conclude that

(9) $\frac{\partial K}{\partial a}(t,s,a) = \{H(t)P^{-1}(a)\}\ast(s).$

This formal argument can be justified by the following result corresponding to the lemma proved in §5.

Lemma. Let $M(t,s)$ be a continuous $n \times n$ matrix function for $a \le s \le 1$, $a \le t \le 1$ and suppose that M has the symmetry property

$$M\ast(t,s) = M(s,t).$$

If

$$\int_a^1 \int_a^1 (M(t,s)y(s),y(t))dsdt = 0$$

for all continuous vector functions y, then $M(s,t) \equiv 0$.

We omit a proof of this lemma because one can be constructed in a way quite similar to that employed in §5. It is also not difficult to verify that the matrix function in (6) has the required symmetry property because the system is self-adjoint.

Combining (7) and (9) we obtain

(10) $\frac{\partial K}{\partial a}(t,s,a) = \frac{\partial K}{\partial s}(t,s,a)\Big|_{s=a} P(a)\frac{\partial K}{\partial t}(t,s,a)\Big|_{t=a}.$

The problem we have just treated does not contain as a special case the problem with the boundary condition $x(1) = 0$. However, only a small change is required to handle this problem by the same method. In the definition of $J(x,y)$ one

can omit the term $-(F(1)B(1)x(1),x(1))$ and for the maximization problem add the constraint $x(1) = 0$. The remainder of the argument is quite similar and the final result obtained, equation (10), is the same.

The approach by way of a change of variable as in $\S 6$ can also be followed in the matrix case. Set

$$x(t) = (I + (1 - t)B)(I + (1 - a)B)^{-1}c + w(t).$$

This transformation has the property that if $x(a) = c$ then $w(a) = 0$; and if $\dot{x}(1) + Bx(1) = 0$, then $\dot{w}(1) + Bw(1) = 0$. Here we must also make the additional assumption that the inverse $(I + (1 - a)B)^{-1}$ exists. Proceeding as in $\S 6$, we obtain the equation

$$\mathfrak{X}(t) = \left[I + (1 - t)B - \int_a^1 K(t,s)\left[Q(s)(I + (1 - s)B)\right.\right.$$

$$\left.\left. - \frac{d}{ds}(P(s))B\right]ds\right]$$

$$\cdot (I + (1 - a)B)^{-1},$$

which allows the derivation of another expression for $\frac{\partial K}{\partial a}$.

12. Integral Equations

Let us now indicate briefly how the same formalism may be applied to integral equations.

The equation

$$(1) \qquad u(x) = v(x) + \lambda \int_a^1 k(x,y)u(y)dy$$

has the solution

$$(2) \qquad u(x) = v(x) + \int_a^1 K(x,y,\lambda)v(y)dy,$$

where $K(x,y,\lambda)$ is the Fredholm resolvent.

If $k(x,y)$ is positive definite, we can consider (1) to be the Euler equation corresponding to the problem of minimizing the quadratic functional

$$(3) \qquad Q(u) = \lambda \int_a^1 \int_a^1 k(x,y)u(x)u(y)dxdy$$

$$+ 2 \int_a^1 u(x)v(x)dx - \int_a^1 u^2(x)dx.$$

Regarding the minimum over u as a function of a and functional of v, we can proceed very much as above. A derivation of the variational equation for Fredholm resolvent is given in [2]. Continuing as in §34, we can derive variational equations for the characteristic values and functions.

REFERENCES

1. Bellman, R., "Functional equations in the theory of dynamic programming—VIII: the variation of Green's functions for the one—dimensional case," Proc. Nat. Acad. Sci. USA, Vol. 43, 1957, pp. 839–841.

2. Bellman, R., "Functional equations in the theory of dynamic programming—VII: a partial differential equation for the Fredholm resolvent," Proc. Amer. Math. Soc., Vol. 8, 1957, pp. 435–440.

3. Bellman, R., "On some applications of dynamic programming to matrix theory," Illinois Journal of Math., Vol. 1, 1957, pp. 297–301.

4. Bellman, R., and H. Osborn, "Dynamic programming and the variation of Green's functions," Jour. Math. and Mech., Vol. 7, 1958, pp. 81–85.

5. Bellman, R., and S. Lehman, "Functional equations in the theory of dynamic programming—IX: imbedding of non—selfadjoint operators in families of self—adjoint operators," (to appear).

6. Miller, K. S., and M. M. Schiffer, "On the Green's functions of ordinary differential systems," Proc. Amer. Math. Soc., Vol. 3, 1952, pp. 433–441.